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Journal of Approximation Theory 132 (2005) 34–41
Journal of Approximation Theory
www.elsevier.com/locate/jat

A local version of the Pawłucki–Pleśniak extension operator

M. Altun, A. Goncharov*
Department of Mathematics, Bilkent University, 06800 Ankara, Turkey

Received 1 December 2003; received in revised form 3 September 2004; accepted in revised form 27 October 2004
Communicated by Manfred Ǒolitschek

Abstract

Using local interpolation of Whitney functions, we generalize the Pawłucki and Pleśniak approach to construct a continuous linear extension operator. We show the continuity of the modified operator in the case of generalized Cantor-type sets without Markov's Property.

MSC: primary 46E10; secondary 41A05; 41A10

Keywords: Whitney functions; Extension operator; Local interpolation; Cantor-type sets; Markov's Property

1. Introduction

For a compact set $K \subset \mathbb{R}^d$, let $\mathcal{E}(K)$ denote the space of Whitney jets on K (see e.g. [24] or [11]). The problem of the existence of an extension operator (here and in what follows it means a continuous linear extension operator) $L : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^d)$ was first considered in [4,13,20,21]. In [22], a topological characterization (*DN* property) for the existence of an extension operator was given. In elaboration of Whitney's method Schmets and Valdivia proved in [19] (see also [7]) that if the extension operator L exists, then one can take a map such that all extensions are analytic on the complement of the compact set. For the extension problem in the classes of ultradifferentiable functions see, for example, [5,17]

* Corresponding author. Fax: +90 312 266 45 79.
E-mail addresses: altun@fen.bilkent.edu.tr (M. Altun), goncha@fen.bilkent.edu.tr (A. Goncharov).

0021-9045/$ - see front matter
doi:10.1016/j.jat.2004.10.013

and the references therein. In [14] (see also [15,18]), Pawłucki and Pleśniak suggested an explicit construction of the extension operator for a rather wide class of compact sets, preserving Markov's inequality. In [8] and later in [9], the compact sets K were presented without Markov's Property, such that the space $\mathcal{E}(K)$ admitted an extension operator. Here, we deal with the generalized Cantor-type sets $K^{(\alpha)}$ that have the extension property for $1 < \alpha < 2$, as it was proved in [9], but are not Markov's sets for any $\alpha > 1$ in accordance with Pleśniak's [16] and Białas's [3] results. The extension operator in [14] was given in the form of a telescoping series containing Lagrange interpolation polynomials with the Fekete–Leja system of knots. This operator is continuous in the Jackson topology τ_J, which is equivalent to the natural topology τ of the space $\mathcal{E}(K)$, provided that the compact set K admits Markov's inequality. Here, following [10], we interpolate the functions from $\mathcal{E}(K^{(\alpha)})$ locally and show that the modified operator is continuous in τ.

2. Jackson topology

For a perfect compact set K on the line, $\mathcal{E}(K)$ denotes the space of all functions f on K extendable to some $F \in C^\infty(\mathbb{R})$. The topology τ of Fréchet space in $\mathcal{E}(K)$ is given by the norms

$$\| f \|_q = |f|_q + \sup\{ |(R_y^q f)^{(k)}(x)| \cdot |x-y|^{k-q};\ x, y \in K, x \neq y,\\ k = 0, 1, \dots, q\},$$

$q = 0, 1, \dots,$ where $|f|_q = \sup\{|f^{(k)}(x)| : x \in K, k \leqslant q\}$ and $R_y^q f(x) = f(x) - T_y^q f(x)$ is the Taylor remainder.

The space $\mathcal{E}(K)$ can be identified with the quotient space $C^\infty(I)/Z$, where I is a closed interval containing K and $Z = \{F \in C^\infty(I) : F|_K \equiv 0\}$. Given $f \in \mathcal{E}(K)$, let $|||\, f \,|||_q = \inf |F|_q^{(I)}$, where the infimum is taken for all possible extensions of f to F and $|F|_q^{(I)}$ denotes the qth norm of F in $C^\infty(I)$. The quotient topology τ_Q, given by the norms $(|||\cdot|||_q)$, is complete; by the open mapping theorem, it is equivalent to the topology τ. Therefore, for any q there exists $r \in \mathbb{N},\ C > 0$ such that

$$|||\, f \,|||_q \leqslant C \,||\, f \,||_r \tag{1}$$

for any $f \in \mathcal{E}(K)$.

Following Zerner [25], Pleśniak [15] introduced in $\mathcal{E}(K)$ the following seminorms:

$$d_{-1}(f) = |f|_0,\ d_0(f) = E_0(f),\ d_k(f) = \sup_{n \geqslant 1} n^k E_n(f)$$

for $k = 1, 2, \dots$. Here, $E_n(f)$ denotes the best approximation to f on K by polynomials of degree at most n. For a perfect set $K \subset \mathbb{R}$ the *Jackson topology* τ_J, given by (d_k), is Hausdorff. By the Jackson theorem (see, e.g. [23]) the topology τ_J is well-defined and is not stronger than τ.

The characterization of analytic functions on a compact set K in terms of (d_k) was considered in [2]; for the spaces of ultradifferentiable functions see [6].

We remark that for any perfect set K, the space $(\mathcal{E}(K), \tau_J)$ has the dominating norm property (see, e.g. [12]):

$$\exists p\ \forall q\ \exists r,\ C > 0: \ d_q^2(f) \leqslant C\ d_p(f)\, d_r(f) \quad \text{for all} \quad f \in \mathcal{E}(K).$$

Indeed, let n_k be such that $d_k(f) = n_k^k\, E_{n_k}(f)$. Then, $d_p(f) \geqslant n_q^p\, E_{n_q}(f)$ and $d_r(f) \geqslant n_q^r\, E_{n_q}(f)$. So we have the desired condition with $r = 2q$.

Tidten proved in [22] that the space $\mathcal{E}(K)$ admits an extension operator if and only if it has the property (DN). Clearly, the completion of the space with the property (DN) also has the dominating norm. Therefore, the Jackson topology is not generally complete. Moreover, it is not complete in the cases of compact sets from [8,9] in spite of the fact that the corresponding spaces have the (DN) property. By Theorem 3.3 in [15], the topologies τ and τ_J coincide for $\mathcal{E}(K)$ if and only if the compact set K satisfies the Markov Property (see [14–18] for the definition) and this is possible if and only if the extension operator, presented in [14,15,18], is continuous in τ_J. We do not know the distribution of the Fekete points for Cantor-type sets, and therefore we cannot check the continuity of the Pawłucki and Pleśniak operator in the natural topology. Instead, following [10], we will interpolate the functions from $\mathcal{E}(K)$ locally.

3. Extension operator for $\mathcal{E}(K^{(\alpha)})$

Let $(l_s)_{s=0}^{\infty}$ be a sequence such that $l_0 = 1,\ 0 < 2l_{s+1} < l_s,\ s \in \mathbb{N}$. Let K be the Cantor set associated with the sequence (l_s), that is, $K = \bigcap_{s=0}^{\infty} E_s$, where $E_0 = I_{1,0} = [0, 1],\ E_s$ is a union of 2^s closed *basic* intervals $I_{j,s}$ of length l_s and E_{s+1} is obtained by deleting the open concentric subinterval of length $l_s - 2l_{s+1}$ from each $I_{j,s}$, $j = 1, 2, \ldots, 2^s$.

Fix $1 < \alpha < 2$ and l_1 with $2l_1^{\alpha-1} < 1$. We will denote by $K^{(\alpha)}$ the Cantor set associated with the sequence (l_n), where $l_0 = 1$ and $l_{n+1} = l_n^{\alpha} = \cdots = l_1^{\alpha^n}$ for $n \geqslant 1$.

In the notations of Arslan et al. [1], we consider the set $K_2^{(\alpha)}$. The construction of the extension operator for the case $K_n^{(\alpha)}$ with $\alpha < n$ is quite similar, so we can restrict ourselves to $n = 2$.

Let us fix $s,\ m \in \mathbb{N}$ and take $N = 2^m - 1$. The interval $I_{1,s}$ covers 2^{m-1} basic intervals of the length l_{s+m-1}. Then $N + 1$ endpoints (x_k) of these intervals give us the interpolating set of the Lagrange interpolation polynomial $L_N(f, x, I_{1,s}) = \sum_{k=1}^{N+1} f(x_k)\, \omega_k(x)$, corresponding to the interval $I_{1,s}$. Here, $\omega_k(x) = \frac{\Omega_{N+1}(x)}{(x-x_k)\Omega'_{N+1}(x_k)}$ with $\Omega_{N+1}(x) = \Pi_{k=1}^{N+1}(x-x_k)$. In the case $2^m < N + 1 < 2^{m+1}$, we use the same procedure as in [10] to include new $N + 1 - 2^m$ endpoints of the basic intervals of the length l_{s+m} in the interpolation set. The polynomials $L_N(f, x, I_{j,s})$, corresponding to other basic intervals, are taken in the same manner.

Given $\delta > 0$, and a compact set E, we take a C^{∞}-function $u(\cdot, \delta, E)$ with the properties: $u(\cdot, \delta, E) \equiv 1$ on E, $u(x, \delta, E) = 0$ for $\mathrm{dist}(x, E) > \delta$ and $|u|_p \leqslant c_p\ \delta^{-p}$, where the constant c_p depends only on p. Let $(c_p) \uparrow$.

Fix $n_s = [s \log_2 \alpha]$ for $s \geqslant \log 4 / \log \alpha$, $n_s = 2$ for the previous values of s and $\delta_{N,s} = l_{s+[\log_2 N]}$ for $N \geqslant 2$. Here $[a]$ denotes the greatest integer in a.

Let $N_s = 2^{n_s} - 1$ and $M_s = 2^{n_{s-1}-1} - 1$ for $s \geqslant 1$, $M_0 = 1$. Consider the operator from [10]

$$\begin{aligned} L(f,x) = {} & L_{M_0}(f,x,I_{1,0})\, u(x,\delta_{M_0+1,0}, I_{1,0} \cap K) \\ & + \sum_{s=0}^{\infty} \left\langle \sum_{j=1}^{2^s} \sum_{N=M_s+1}^{N_s} [L_N(f,x,I_{j,s}) - L_{N-1}(f,x,I_{j,s})] \right. \\ & \times u(x,\delta_{N,s}, I_{j,s} \cap K) \\ & + \sum_{j=1}^{2^{s+1}} [L_{M_{s+1}}(f,x,I_{j,s+1}) - L_{N_s}(f,x,I_{[\frac{j+1}{2}],s})] \\ & \left. \times\ u(x,\delta_{N_s,s}, I_{j,s+1} \cap K) \right\rangle . \end{aligned}$$

We call the sums $\sum_{N=M_s+1}^{N_s} \cdots$ the *accumulation sums*. For fixed *j* (without loss of generality let $j = 1$) represent the term in the last sum in the telescoping form

$$- \sum_{N=2^{n_s-1}}^{2^{n_s}-1} [L_N(f,x,I_{1,s}) - L_{N-1}(f,x,I_{1,s})]\, u(x, l_{s+n_s-1}, I_{1,s+1} \cap K)$$

and will call this the *transition sum*. Here, the interpolation set for the polynomial $L_N(f,x, I_{1,s})$ consists of all endpoints of the basic subintervals of length l_{s+n_s-1} on $I_{1,s+1}$ and some endpoints (from 0 for $N = 2^{n_s-1} - 1$ to all for $N = 2^{n_s} - 1$) of basic subintervals of the same length on $I_{2,s+1}$.

Clearly, the operator L is linear. Let us show that it extends the functions from $\mathcal{E}(K^{(\alpha)})$.

Lemma 1. *For any* $f \in \mathcal{E}(K^{(\alpha)})$ *and* $x \in K^{(\alpha)}$*, we have* $L(f,x) = f(x)$.

Proof. By the telescoping effect

$$L(f,x) = \lim_{s\to\infty} L_{M_s}(f,x,I_{j,s}), \tag{2}$$

where $j = j(s)$ is chosen in such a way that $x \in I_{j,s}$.

We will denote temporarily $n_{s-1} - 1$ by n. Then $M_s = 2^n - 1$. Arguing as in [10], for any $q,\ 1 \leqslant q \leqslant M_s$, we have the bound

$$|\, L_{M_s}(f,x,I_{j,s}) - f(x)| \leqslant ||\, f\, ||_q \sum_{k=1}^{2^n} |\, x - x_k|^q\, |\, \omega_k(x)\, |. \tag{3}$$

For the denominator of $|\,\omega_k(x)\,|$ we get

$$\begin{aligned} & |\, x_k - x_1| \cdots |\, x_k - x_{k-1}| \cdot |\, x_k - x_{k+1}| \cdots |\, x_k - x_{M_s+1}| \\ & \geqslant l_{n+s-1}\, (l_{n+s-2} - 2\, l_{n+s-1})^2 \cdot (l_{n+s-3} - 2\, l_{n+s-2})^4 \cdots (l_s - 2\, l_{s+1})^{2^{n-1}} \\ & = l_{n+s-1} \cdot l_{n+s-2}^2 \cdots l_s^{2^{n-1}} \cdot A, \end{aligned}$$

where $A = \Pi_{k=1}^{n-1} (1 - 2\, \frac{l_{s+k}}{l_{s+k-1}})^{2^{n-k}}$.

Clearly, $\ln A > -\sum_{k=1}^{n-1} 2^{n-k+2} \frac{l_{s+k}}{l_{s+k-1}}$ for large enough s. Since $\frac{l_{s+k}}{l_{s+k-1}} < \frac{l_{s+k-1}}{l_{s+k-2}}$ and $2^n \leqslant \frac{1}{2}\,\alpha^{s-1}$, we have $\ln A > -2^{n+2}\, l_s^{\alpha-1} > -1$.

On the other hand, the numerator of $|\,\omega_k(x)\,|$ multiplied by $|\,x - x_k|^{\,q}$ gives the bound

$$|\,x - x_k|^{\,q-1}\,\Pi_1^{2^n}|\,x - x_k| \leqslant l_s^{q-1} \cdot l_{n+s} \cdot l_{n+s-1} \cdot l_{n+s-2}^2 \cdots l_s^{\,2^{n-1}}.$$

Hence, the sum in (3) may be estimated from above by $e\ 2^n\ l_{n+s}\, l_s^{q-1}$, which approaches 0 as s becomes large. Therefore, the limit in (2) exists and equals $f(x)$. □

4. Continuity of the operator L

Theorem 1. *Let* $1 < \alpha < 2$. *The operator* $L : \mathcal{E}(K^{(\alpha)}) \longrightarrow C^\infty(\mathbb{R})$, *given in Section* 3, *is a continuous linear extension operator.*

Proof. Let us prove that the series representing the operator L uniformly converges together with any of its derivatives.

For any $p \in \mathbb{N}$, let $q = 2^v - 1$ be such that $(2/\alpha)^v > p + 4$. Given q let s_0 satisfy the following conditions: $s_0 \geqslant 2v + 3$ and $\alpha^m \geqslant m$ for $m \geqslant n_{s_0-1}$.

Suppose the points $(x_k)_1^{N+1}$ are arranged in ascending order. For the divided difference $[x_1, \ldots, x_{N+1}]f$, we have the following bound from [10]:

$$|\ [x_1, \ldots, x_{N+1}]f\ | \leqslant 2^{N-q}\ |||f|||_q\ \left(\min \Pi_{m=1}^{N-q}|\,x_{a(m)} - x_{b(m)}|\right)^{-1}, \tag{4}$$

where min is taken over all $1 \leqslant j \leqslant N + 1 - q$ and all possible chains of strict embeddings $[x_{a(0)}, \ldots, x_{b(0)}] \subset [x_{a(1)}, \ldots, x_{b(1)}] \subset \cdots \subset [x_{a(N-q)}, \ldots, x_{b(N-q)}]$ with $a(0) = j,\ b(0) = j + q, \ldots, a(N-q) = 1,\ b(N-q) = N + 1$. Here, given $a(k),\ b(k)$, we take $a(k+1) = a(k),\ b(k+1) = b(k) + 1$ or $a(k+1) = a(k) - 1,\ b(k+1) = b(k)$. The length of the first interval in the chain is not included in the product in (4), which we denote in the sequel by Π.

For $s \geqslant s_0$ and for any $j \leqslant 2^s$ we consider the corresponding term of the accumulation sum. By the Newton form of interpolation operator we get

$$L_N(f, x, I_{j,s}) - L_{N-1}(f, x, I_{j,s}) = [x_1, \ldots, x_{N+1}]f \cdot \Omega_N(x),$$

where $\Omega_N(x) = \Pi_1^N(x - y_k)$ with the set $(y_k)_1^N$ consisting of all points $(x_k)_1^{N+1}$ except one.

Thus, we need to estimate $|\ [x_1, \ldots, x_{N+1}]f\ |\ \cdot |(\,\Omega_N \cdot u(x, \delta_{N,\,s}, I_{j,s} \cap K))^{(p)}|$ from above. Here $M_s + 1 \leqslant N \leqslant N_s$, that is $2^{m-1} \leqslant N < 2^m$ for some $m = n_{s-1}, \ldots, n_s$ and $\delta_{N,\,s} = l_{s+m-1}$. The interpolation set $(x_k)_1^{N+1}$ consists of all endpoints of the basic intervals of length l_{s+m-2} (inside the interval $I_{j,s}$) and some endpoints (possibly all for $N = 2^m - 1$) of the basic intervals of length l_{s+m-1}. For simplicity we take $j = 1$. In this case, $x_1 = 0,\ x_2 = l_{s+m-1},\ x_3 = l_{s+m-2} - l_{s+m-1}$ or $x_3 = l_{s+m-2}$, etc.

Since $\mathrm{dist}(x, I_{1,s} \cap K) \leqslant l_{s+m-1}$, we get

$$|\,\Omega_N^{(i)}(x)\,| \leqslant \frac{N!}{(N-i)!}\,\Pi_{k=i+1}^N(l_{s+m-1} + y_k).$$

Therefore, $|(\Omega_N \cdot u)^{(p)}| \leqslant \sum_{i=0}^{p} \binom{p}{i} c_{p-i}\, l_{s+m-1}^{i-p}\, N^i\, \Pi_{k=i+1}^{N}(l_{s+m-1}+y_k) \leqslant 2^p\, c_p\, l_{s+m-1}^{-p}$ $\Pi_{k=1}^{N}(l_{s+m-1}+y_k)$. $\max_{i\leqslant p} B_i$, with $B_0=1,\ B_1=N,\ B_2=N^2/2, \ldots, B_i = N^2/2 \cdot (N\, l_{s+m-1})^{i-2}\ (l_{s+m-1}+y_3)^{-1}\cdots(l_{s+m-1}+y_i)^{-1}$ for $i \geqslant 3$.

To estimate B_3, we note that $l_{s+m-1}+y_3 \geqslant l_{s+m-2},\ N\, l_{s+m-1} < 2^m\, l_{s+m-2}^{\alpha} \leqslant l_{s+m-2}$ since $2^m\, l_{s+m-2}^{\alpha-1} = 2^m\, l_{s-2}^{(\alpha-1)\alpha^m} < 2^m\, l_1^{(\alpha-1)\alpha^m} < 2^m (\frac{1}{2})^{\alpha^m} \leqslant 1$, due to the choice of s_0. Therefore, B_3, and all B_i for $i > 3$, are less than B_2. On the other hand, $l_{s+m-1}+y_k < y_{k+1},\ k \leqslant N-1$, as l_{s+m-1} is a mesh of the net $(y_k)_1^N$ and $l_{s+m-1}+y_N < 2l_s$. This implies that

$$|(\Omega_N \cdot u)^{(p)}| \leqslant 2^p\, c_p\, N^2\, l_{s+m-1}^{-p}\, l_s\, \Pi_{k=2}^{N} y_k \leqslant 2^p\, c_p\, N^2\, l_{s+m-1}^{-p-1}\, l_s\, \Pi_{k=2}^{N+1} x_k. \tag{5}$$

To apply (4), for $1 \leqslant j \leqslant N+1-q$ we consider $q+1$ consecutive points $(x_{j+k})_{k=0}^{q}$ from $(x_k)_1^{N+1}$. Every interval of the length l_{s+k} contains from $2^{m-k-1}+1$ to 2^{m-k} points x_k. Therefore, the interval of the length $l_{s+m-v-1}$ contains more than $q+1$ points. In order to minimize the product Π, we have to include intervals containing large gaps in the set $K^{(\alpha)}$ in the chain $[x_j, \ldots, x_{j+q}] \subset \cdots \subset [x_1, \ldots, x_{N+1}]$ as late as possible, that is all $q+1$ points must belong to $I_{j,s+m-v-1}$ for some j. By the symmetry of the set $K^{(\alpha)}$, we can again take $j=1$. The interval of the length l_{s+m-v} contains at most 2^v points, whence for any choice of $q+1$ points in succession, all values that make up the product Π are not smaller than the length of the gap $h_{s+m-v-1} := l_{s+m-v-1} - 2\, l_{s+m-v}$. Therefore, $\Pi \geqslant h_{s+m-v-1}^{J-q-1}\, \Pi_{J+1}^{N+1} x_k$, where J is the number of points x_k on $I_{1,s+m-v-1}$. Since $J \leqslant 2^{v+1}$, we have $J-q-1 \leqslant 2^v$. Further,

$$\frac{x_{q+2}\cdots x_J}{h_{s+m-v-1}^{J-q-1}} \leqslant \left(\frac{l_{s+m-v-1}}{l_{s+m-v-1} - 2\, l_{s+m-v}}\right)^{2^v} < \exp(2^v\, 4 l_{s+m-v-1}^{\alpha-1}). \tag{6}$$

Since $l_{s+m-v-1}^{\alpha-1} = l_1^{(\alpha-1)(s+m-v-2)} < 2^{-s+v}$, we see that the fraction above is smaller than 2, due to the choice of s_0. It follows that $\Pi \geqslant \frac{1}{2}\Pi_{q+2}^{N+1} x_k$ and $|[x_1, \ldots, x_{N+1}]f| \leqslant 2^{N-q-1}\, |||f|||_q\ (x_{q+2}\cdots x_{N+1})^{-1}$.

Combining this with (5) we have

$$|\,[x_1, \ldots, x_{N+1}]f\,| \cdot |(\Omega_N \cdot u)^{(p)}| \leqslant c_p\, N^2\, 2^N\, l_s\ l_{s+m-1}^{-p-1}\, \Pi_{k=2}^{q+1} x_k\, |||f|||_q.$$

Our next goal is to evaluate $\Pi_{k=2}^{q+1} x_k$ in terms of l_{s+m-1}. Estimating roughly all $x_k,\ k > 2$ that are not endpoints of the basic intervals of length l_{s+m-2}, from above by $l_{s+m-v-1}$, we get

$$\Pi_{k=2}^{q+1} x_k \leqslant l_{s+m-1}\, l_{s+m-2}\, l_{s+m-3}^2 \cdots l_{s+m-v}^{2^{v-2}}\, l_{s+m-v-1}^{2^{v-1}-1} = l_{s+m-1}^{\kappa}$$

with $\kappa = 1 + \frac{1}{\alpha} + \frac{2}{\alpha^2} + \cdots + \frac{2^{v-1}}{\alpha^v} - \frac{1}{\alpha^v} > (2/\alpha)^v - 1$.

Therefore,

$$|\,[x_1, \ldots, x_{N+1}]f\,| \cdot |(\Omega_N \cdot u)^{(p)}| \leqslant c_p\, N^2\, 2^N\ l_{s+m-1}^2\, |||f|||_q,$$

since $\kappa + \alpha^{-m+1} - p - 1 > 2$, due to the choice of q.

Here, $2^N\ l_{s+m-1} < 2^{2^m}\ l_1^{\alpha^{s+m-2}} < 2^{2^{n_s}-\alpha^s} \leqslant 1$, as $m \geqslant 2$ and $l_1 < \frac{1}{2}$. The accumulation sum contains $N_s - M_s < N_s$ terms. Therefore,

$$\left| \left(\sum_{N=M_s+1}^{N_s} \cdots \right)^{(p)} \right| \leqslant c_p\, N_s^3\ l_s\, |||f|||_q,$$

which is a term of the series convergent with respect to s, as is easy to see. We neglect the sum with respect to j, because for fixed x, at most one term of this sum does not vanish.

The same proof works for the terms of the transition sums. This sum does not vanish only for x at a short distance to $I_{1,s+1} \cap K$. For this reason, the arguments of the estimation of $|\,\Omega_N^{(i)}(x)|$ remain valid. On the other hand, if we want to minimize the product of the lengths of intervals, constituting the chain $[x_j, \ldots, x_{j+q}] \subset \cdots \subset [x_1, \ldots, x_{N+1}]$, then we have to take $x_j, \ldots, x_{j+q}$ in the interval $I_{1,s+1}$. Thus we have the bound (6). The rest of the proof runs as before. Taking into account (1), we see that the operator L is well-defined and continuous. □

Remark. It is a simple matter to find a sequence of functions that converges in the Jackson topology and diverges in τ. It is interesting that the same sequence can destroy the Markov inequality. Given $s \in \mathbb{N}$, let $N = 2^s$ and $P_N(x) = (l_{s-1} \cdot l_{s-2}^2 \cdots l_0^{2^{s-1}})^{-1} \prod_{j=1}^N (x - c_{j,s})$, where $c_{j,s}$ is a midpoint of the interval $I_{j,s}$. Then $\frac{1}{s}\ \ln(|P_N'(0)|/|P_N|_0) \to \infty$ as $s \to \infty$, contrary to the Markov property. On the other hand, $E_n(P_N) \leqslant |P_N|_0$ for $n < N$. Then, for any k we get $d_k(P_N) \leqslant N^k\, |P_N|_0 \leqslant 2^{s\,k}\, l_s \to 0$ as $s \to \infty$. But $P_N'(0) \not\to 0$, so the sequence (P_N) diverges in the natural topology of the space $\mathcal{E}(K^{(\alpha)})$.

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